Name:________________________________

Lessons 3 and 4
Integration by Substitution
Math 16020

1 Integration Rule Resulting from the Chain Rule

In the last class, we frequently used the sum and difference rules when integrating.

(a) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

(b) $\int (f(x) - g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$

Unfortunately, there are no simple integration rules for multiplication and division.

Example 1. Evaluate

(a) $\int (x^2)(x^3)\, dx = \int x^5 dx = \frac{x^6}{6} + C$

not equal

(b) $\int (x^2)\, dx \int (x^3)\, dx = \left(\frac{x^3}{3} + C_1\right)\left(\frac{x^4}{4} + C_2\right) = \frac{x^7}{12} + C_2\frac{x^3}{3} + C_1\frac{x^4}{4} + C_1C_2$

(c) What are you supposed to learn from this example?

No simple product rule for integrals

Although there is not a simple integration formula for products, every derivative rule is a associated with an integral rule. The Chain Rule for derivatives does lead to an integration formula that will help us to integrate some special products.

Theorem 2 (The Chain Rule)**.** If g is differentiable at x and f is differentiable at $g(x)$, then

$$[f(g(x))]' = f'(g(x))g'(x)$$

This derivative rule leads to the integration formula

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C$$

If we make the substitution

$$u = g(x),$$ → u is often inside function

then

$$\frac{du}{dx} = g'(x) \implies du = g'(x)\,dx$$

so we can rewrite our integration formula as

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C$$

Example 3. $\int (3x^2)(x^3+10)^{15}\,dx = \int u^{15}\,du = \frac{u^{16}}{16} + C$

inside fctn

$= \frac{(x^3+10)^{16}}{16} + C$

$u = x^3 + 10$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

2 Substitution Examples

As we do these examples, here are some things to keep in mind.

$\frac{a}{b} = a \cdot \frac{1}{b}$

1. Substitution might be a good tool to try if you are working with a product or a quotient (which is a product in disguise).
2. If you are going to use substitution, the function you are trying to integrate *often* contains a function **and its derivative** - *up to a constant multiple.*
3. The function you choose for u is *often* inside of another function.
4. If you set $u = g(x)$ so that $du = g'(x)dx$, then ALL of $g'(x)$ is multiplied by dx. For example, if $u = x^2 + 3x$, then

$$du = (2x + 3)dx$$

It is **NOT** correct to write

$$du = \cancel{2x + 3dx}$$

5. After you find du, you can multiply or divide both sides of your equation **by a constant** to make the substitution easier.
6. After you make the substitution, all of the variables in your new integral should be u's. Your integral should **ONLY CONTAIN ONE VARIABLE** - and it should be easier to solve.
7. When you are finding an indefinite integral (no limits) using substitution, your last step is **BACK SUBSTITUTION**. Your **final answer** should be in the **ORIGINAL VARIABLE**.

Example 4. $\int 3x\sqrt{x^2+5}\,dx$ $= \int 3\sqrt{u}\,\frac{1}{2}du = \int \frac{3}{2}\sqrt{u}\,du = \int \frac{3}{2}u^{1/2}\,du$

$u = x^2 + 5$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{3}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= u^{3/2} + C = (x^2+5)^{3/2} + C$

Example 5. $\int (3x+4)^{17}dx$ $= \int (u)^{17}\cdot\frac{1}{3}du = \frac{1}{3}\frac{u^{18}}{18} + C$

$= \frac{u^{18}}{54} + C$

$= \frac{(3x+4)^{18}}{54} + C$

$u = 3x + 4$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

3 Substitution with Definite Integrals

Whenever you are using substitution to solve a definite integral (with limits), you must **USE YOUR SUBSTITUTION EQUATION** to **CHANGE THE LIMITS TO** u**-LIMITS**.

Example 6. $\int_0^1 2x^6 e^{x^7+3}\,dx = \int_3^4 2e^u \cdot \frac{1}{7}du = \frac{2}{7}e^u \Big|_3^4 = \frac{2}{7}e^4 - \frac{2}{7}e^3$

$= \frac{2}{7}(e^4 - e^3)$

$u = x^7 + 3$

$du = 7x^6\,dx$

$\frac{1}{7}du = x^6\,dx$

when $x=0$, $u = 0^7 + 3 = 3$

when $x=1$, $u = 1^7 + 3 = 4$

Example 7. $\int_0^{\frac{\pi}{3}} \sin(x)\cos^4(x)\,dx = \int_0^{\pi/3} \sin(x)(\cos(x))^4\,dx$

$= \int_1^{1/2} (u)^4 \cdot -du$

$= \frac{-u^5}{5}\Big|_1^{1/2} = \frac{-(1/2)^5}{5} - \frac{-1^5}{5}$

$= \frac{-(1/2)^5}{5} + \frac{1^5}{5} = \frac{-1/32}{5} + \frac{1}{5}$

$= \frac{-1}{160} + \frac{1}{5} = \frac{31}{160}$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$x = 0 \Rightarrow u = \cos(0) = 1$

$x = \frac{\pi}{3} \Rightarrow u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

4 More examples

Example 8. A function $f(x)$ has tangent line slope $\underbrace{x\sqrt{x-2}}_{f'(x)}$ for all $x > 2$. The graph of f passes through the point $\left(3, \frac{9}{15}\right)$. Find a formula for $f(x)$.

$$f'(x) = x\sqrt{x-2}$$

$$f(x) = \int x\sqrt{x-2}\,dx = \int (u+2)\underbrace{\sqrt{u}}_{u^{1/2}}\,du = \int \left(u^{3/2} + 2u^{1/2}\right) du$$

$u = x-2 \Rightarrow u+2 = x$

$du = 1\,dx$

$$= \frac{2}{5}u^{5/2} + 2\cdot\frac{2}{3}u^{3/2} + C$$

$$f(x) = \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$$

$$\frac{9}{15} = \frac{2}{5}(3-2)^{5/2} + \frac{4}{3}(3-2)^{3/2} + C$$

$$\frac{9}{15} = \frac{2}{5} + \frac{4}{3} + C$$

$$\frac{9}{15} = \frac{6}{15} + \frac{20}{15} + C$$

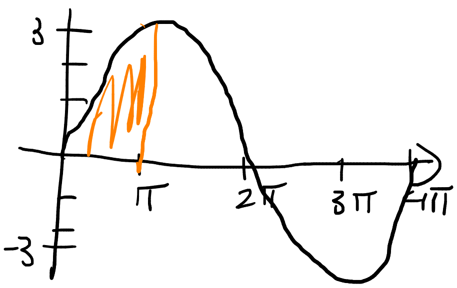
$$\frac{9}{15} = \frac{26}{15} + C$$

$$C = \frac{9}{15} - \frac{26}{15} = \frac{-17}{15}$$

Ans:

$$f(x) = \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} - \frac{17}{15}$$

Example 9. Find the area under the curve $y = 3\sin(0.5x)$ from $x = 0$ to $x = \pi$.



$$A = \int_0^{\pi} 3\sin(0.5x)\,dx = \int_0^{\pi/2} 3\sin(u)\,2\,du$$

$u = 0.5x$

$du = 0.5\,dx$

$\frac{1}{0.5}du = dx$

$2\,du = dx$

when $x = 0$

$u = 0.5(0) = 0$

when $x = \pi$

$u = 0.5\pi = \frac{\pi}{2}$

$$= 6(-\cos(u))\Big|_0^{\pi/2}$$

$$= -6\cos\left(\frac{\pi}{2}\right) + 6\cos(0)$$

$$= -6(0) + 6(1)$$

$$= 6$$

Answer

Example 10. $\int_0^{12} \frac{x}{\sqrt{x+4}}\,dx$ $= \int_4^{16} \frac{u-4}{\sqrt{u}}\,du = \int_4^{16}\left(\frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\right)du$

$u = x+4$ $\quad x=0 \Rightarrow u = 0+4 = 4$

$du = dx$ $\quad x=12 \Rightarrow u = 12+4 = 16$

$u = x+4$

$u-4 = x$

$= \int_4^{16}\left(u^{1/2} - 4u^{-1/2}\right)du$

$= \frac{2}{3}u^{3/2} - 4\cdot 2\,u^{1/2}\Big|_4^{16}$

$= \left(\frac{2}{3}(16)^{3/2} - 8(16)^{1/4}\right)$

$- \left(\frac{2}{3}(4)^{3/2} - 8(4)^{1/2}\right)$

$= \left(\frac{2}{3}(64) - 8(4)\right) - \left(\frac{2}{3}(8) - 8(2)\right)$

$= \frac{64}{3}$

Example 11. The area under the curve $3e^{0.2x}$ on the interval $0 \le x \le a$ is 45. What is a?

$$\boxed{\int_0^a 3e^{0.2x}\,dx = 45}$$

$3e^{0.2x} = 3\exp(0.2x)$

$u = 0.2x$

$u = \frac{1}{5}x$

$du = \frac{1}{5}dx$

$5du = dx$

$x = 0 \Rightarrow u = \frac{1}{5}(0) = 0$

$x = a \Rightarrow u = \frac{1}{5}(a) = \frac{a}{5}$

$\int_0^{a/5} 3e^u \cdot 5\,du = 45$

$15e^u\Big|_0^{a/5} = 45$

$15e^{a/5} - 15\underbrace{e^0}_{1} = 45$

$15e^{a/5} - 15 = 45$
$\quad\;\; +15 \quad\; +15$

$\frac{15e^{a/5}}{15} = \frac{60}{15}$

$e^{a/5} = 4$

$a/5 = \ln 14)$

$a = 5\ln(4)$